Circuit double cover conjecture
an introduction

CQ Zhang
West Virginia University
Circuit double cover conjecture:
(Tutte ≤ 70’s, Szekeres 1973, Itai and Rodeh 1978, Seymour 1979)
Every bridgeless graph has a family of circuits that covers every edge precisely twice.
Circuit double cover conjecture:
(Tutte ≤ 70's, Szekeres 1973, Itai and Rodeh 1978, Seymour 1979)
Every bridgeless graph has a family of circuits that covers every edge precisely twice.
Circuit double cover conjecture:
(Tutte ≤ 70’s, Szekeres 1973, Itai and Rodeh 1978, Seymour 1979)
*Every bridgeless graph has a family of circuits that covers every edge precisely twice.*

Circuit is a 2-regular connected graph.
Bridge is a cut-edge (separating two parts of a graph, not contained in any circuit).
Circuit double cover conjecture:
(Tutte $\leq 70$’s, Szekeres 1973, Itai and Rodeh 1978, Seymour 1979)

Every bridgeless graph has a family of circuits that covers every edge precisely twice.

Circuit is a 2-regular connected graph.
Circuit double cover conjecture:
(Tutte ≤ 70’s, Szekeres 1973, Itai and Rodeh 1978, Seymour 1979)
Every bridgeless graph has a family of circuits that covers every edge precisely twice.

\[ \text{Circuit} \text{ is a 2-regular connected graph.} \]
\[ \text{Bridge is a cut-edge (separating two parts of a graph, not contained in any circuit).} \]
Wrong proof # 1

It is easy!!!!
Wrong proof # 1

It is easy!!!! Just take the collection of the boundaries of all faces.
Wrong proof # 1

It is easy!!!! Just take the collection of the boundaries of all faces.

- **WRONG!**
  - Not every graph is planar.
Wrong proof # 1

It is easy!!!! Just take the collection of the boundaries of all faces.

- **WRONG!**
  Not every graph is planar.
- How about a graph embedded on a surface (2-manifold)?
Wrong proof # 1

It is easy!!!! Just take the collection of the boundaries of all faces.

- **WRONG!**
  - Not every graph is planar.
  - How about a graph embedded on a surface (2-manifold)?
  - Although every graph can be embedded on some surface, it is not sure that every face is bounded by a circuit.
Wrong proof # 1

It is easy!!!! Just take the collection of the boundaries of all faces.

- **WRONG!**
  - Not every graph is planar.
  - How about a graph embedded on a surface (2-manifold)?
  - Although every graph can be embedded on some surface, it is not sure that every face is bounded by a circuit.

- **Strong embedding conjecture.** (a stronger topology conjecture) *every bridgeless graph can be embedded on some surface such that every face is bounded by a circuit.*
Wrong proof # 2

$2G$ is eulerian
Wrong proof # 2

$2G$ is eulerian
$\Rightarrow 2G$ has a circuit decomposition $\mathcal{F}$.

$\Rightarrow F$ is a circuit double cover of $G$.

WRONG!!

$F$ may contain digons (parallel edges, circuit of length 2).

And a digon in $2G$ does not represent a circuit of $G$. 
Wrong proof # 2

$2G$ is eulerian
$\Rightarrow 2G$ has a circuit decomposition $\mathcal{F}$.
$\Rightarrow \mathcal{F}$ is a circuit double cover of $G$

WRONG!!

$F$ may contain digons (parallel edges, circuit of length 2).
And a digon in $2G$ does not represent a circuit of $G$. 
2G is eulerian
⇒ 2G has a circuit decomposition $F$.
⇒ $F$ is a circuit double cover of $G$

WRONG!!
$F$ may contain digons (parallel edges, circuit of length 2).
And a digon in 2G does not represent a circuit of $G$. 
Circuit double cover conjecture:
Every bridgeless graph has a family of circuits that covers every edge precisely twice.
Circuit double cover conjecture:
Every bridgeless graph has a family of circuits that covers every edge precisely twice.

CDC conjecture is true for

...
Circuit double cover conjecture:
*Every bridgeless graph has a family of circuits that covers every edge precisely twice.*

**CDC conjecture is true for**
- planar graphs
Circuit double cover conjecture:
Every bridgeless graph has a family of circuits that covers every edge precisely twice.

CDC conjecture is true for
- planar graphs
- graphs with strong 2-cell embedding on some surfaces
Circuit double cover conjecture: 
Every bridgeless graph has a family of circuits that covers every edge precisely twice.

CDC conjecture is true for
- planar graphs
- graphs with strong 2-cell embedding on some surfaces
- 3-edge-colorable cubic graphs,
Circuit double cover conjecture:
Every bridgeless graph has a family of circuits that covers every edge precisely twice.

CDC conjecture is true for
- planar graphs
- graphs with strong 2-cell embedding on some surfaces
- 3-edge-colorable cubic graphs,
- etc. (more survey later)
In most cases, we only consider **CUBIC graphs**
In most cases, we only consider **CUBIC graphs** since a smallest counterexample to CDC is
In most cases, we only consider **CUBIC graphs** since a smallest counterexample to CDC is (1) **cubic** (by Fleischner’s splitting lemma),
In most cases, we only consider **CUBIC graphs** since a smallest counterexample to CDC is (1) cubic (by Fleischner’s splitting lemma), (2) cyclically 4-edge-connected, etc.
Vertex splitting

Figure: splitting two edges away from $v$ (Fleischner's lemma)

The resulting graph remains bridgeless.
Figure: CDC of $G$ constructed from $G_1$ and $G_2$.
Figure: CDC of $G$ constructed from $G_1$ and $G_2$
3-edge-coloring

3-edge-coloring of cubic graphs is an extensively studied subject in graph theory.
3-edge-coloring of cubic graphs is an extensively studied subject in graph theory.

For planar graphs:

3-edge-coloring of all bridgeless cubic planar graphs
3-edge-coloring of cubic graphs is an extensively studied subject in graph theory.
For planar graphs:
3-edge-coloring of all bridgeless cubic planar graphs

\[ \iff \]
3-edge-coloring

3-edge-coloring of cubic graphs is an extensively studied subject in graph theory.
For planar graphs:
3-edge-coloring of all bridgeless cubic planar graphs

⇔

Map 4-coloring theorem.
Let $G$ be a cubic graph. The following are equivalent statements:

1. $G$ is 3-edge-colorable;
2. $G$ has three even subgraphs $\{F_1, F_2, F_3\}$ covering every edge precisely twice;
3. $G$ has four even subgraphs $\{F_1, F_2, F_3, F_4\}$ covering every edge precisely twice.

A subgraph $H$ is even iff the degree of every vertex is even.
Let $G$ be a cubic graph. The following are equivalent statements:

(1) $G$ is 3-edge-colorable;

(2) $G$ has three even subgraphs $\{F_1, F_2, F_3\}$ covering every edge precisely twice.

(3) $G$ has four even subgraphs $\{F_1, F_2, F_3, F_4\}$ covering every edge precisely twice.
Let $G$ be a cubic graph. The following are equivalent statements:

(1) $G$ is 3-edge-colorable;

(2) $G$ has three even subgraphs $\{F_1, F_2, F_3\}$ covering every edge precisely twice.

(3) $G$ has four even subgraphs $\{F_1, F_2, F_3, F_4\}$ covering every edge precisely twice.
Let $G$ be a cubic graph. The following are equivalent statements:

1. $G$ is 3-edge-colorable;
2. $G$ has three even subgraphs $\{F_1, F_2, F_3\}$ covering every edge precisely twice.
3. $G$ has four even subgraphs $\{F_1, F_2, F_3, F_4\}$ covering every edge precisely twice.
Let $G$ be a cubic graph. The following are equivalent statements:

(1) $G$ is 3-edge-colorable;
(2) $G$ has three even subgraphs $\{F_1, F_2, F_3\}$ covering every edge precisely twice.
(3) $G$ has four even subgraphs $\{F_1, F_2, F_3, F_4\}$ covering every edge precisely twice.

A subgraph $H$ is even iff the degree of every vertex is even.
Let $G$ be a cubic graph. The following are equivalent statements:

(1) $G$ is 3-edge-colorable;
(2) $G$ has **three** even subgraphs $\{F_1, F_2, F_3\}$ covering every edge precisely twice.
(3) $G$ has **four** even subgraphs $\{F_1, F_2, F_3, F_4\}$ covering every edge precisely twice.

*a subgraph $H$ is even iff the degree of every vertex is even*

(union of circuits)
Proof of (1) $\Rightarrow$ (2)

Let $\{F_1, F_2, F_3\}$ be a 3-even-subgraph double cover.

Color each even subgraph:
- $F_1$: $\leftarrow$ red
- $F_2$: $\leftarrow$ blue
- $F_3$: $\leftarrow$ yellow

Then $c: E(G) \mapsto \{\text{purple}, \text{green}, \text{orange}\}$
Even subgraph double cover and 3-edge-coloring

Proof of (1) ⇒ (2)
“3-edge-colorable” ⇒ “3-even-subgraph double cover”.
\[ c : E(G) \mapsto \{ \text{red}, \text{blue}, \text{yellow} \} \]
Proof of (1) ⇒ (2)

“3-edge-colorable” ⇒ “3-even-subgraph double cover”.

c : E(G) ⟷ {red, blue, yellow}

\{Red ∪ Blue, Red ∪ Yellow, Blue ∪ Yellow\}
Proof of $(1) \Rightarrow (2)$

"3-edge-colorable" $\Rightarrow$ "3-even-subgraph double cover".

$c : E(G) \mapsto \{\text{red, blue, yellow}\}$

$\{\text{Red} \cup \text{Blue}, \text{Red} \cup \text{Yellow}, \text{Blue} \cup \text{Yellow}\}$

Proof of $(2) \Rightarrow (1)$
Proof of (1) $\Rightarrow$ (2)

“3-edge-colorable” $\Rightarrow$ “3-even-subgraph double cover”.

c : $E(G)$ $\mapsto$ \{red, blue, yellow\}
\{Red $\cup$ Blue, Red $\cup$ Yellow, Blue $\cup$ Yellow\}

Proof of (2) $\Rightarrow$ (1)

“3-even-subgraph double cover” $\Rightarrow$ “3-edge-colorable”.
Proof of (1) $\Rightarrow$ (2)

"3-edge-colorable" $\Rightarrow$ "3-even-subgraph double cover".

$c : E(G) \mapsto \{\text{red, blue, yellow}\}$

\{Red \cup Blue, Red \cup Yellow, Blue \cup Yellow\}

Proof of (2) $\Rightarrow$ (1)

"3-even-subgraph double cover" $\Rightarrow$ "3-edge-colorable".

Let \{\(F_1, F_2, F_3\)\} be a 3-even-subgraph double cover.
Even subgraph double cover and 3-edge-coloring

Proof of (1) ⇒ (2)
“3-edge-colorable” ⇒ “3-even-subgraph double cover”.
c : E(G) ↦ {red, blue, yellow}
{Red ∪ Blue, Red ∪ Yellow, Blue ∪ Yellow}

Proof of (2) ⇒ (1)
“3-even-subgraph double cover” ⇒ “3-edge-colorable”.
Let \{F_1, F_2, F_3\} be a 3-even-subgraph double cover. Color each even subgraph:
F_1 :← red
F_2 :← blue
F_3 :← yellow.
Then c : E(G) ↦ {purple, green, orange}
How about 5-even subgraph double cover?

Equivalent statements:

(1) 3-edge-colorable;
(2) 3-even subgraph double cover;
(3) 4-even subgraph double cover.

Can we extend it to 5-even subgraph cover?

5-even subgraph double cover is NOT equivalent to 3-edge-coloring.

The Petersen graph is not 3-edge-colorable, but does have a 5-even subgraph double cover.

Conjecture (Jaeger) Every bridgeless graph has a 5-even subgraph double cover.
How about $5$-even subgraph double cover?

Equivalent statements:
(1) $3$-edge-colorable;

$5$-even subgraph double cover is NOT equivalent to $3$-edge-coloring.

The Petersen graph is not $3$-edge-colorable, but does have a $5$-even subgraph double cover.

Conjecture (Jaeger)
Every bridgeless graph has a $5$-even subgraph double cover.
How about 5-even subgraph double cover?

Equivalent statements:
(1) 3-edge-colorable;
(2) 3-even subgraph double cover;
How about 5-even subgraph double cover?

**Equivalent statements:**

1. 3-edge-colorable;
2. 3-even subgraph double cover;
3. 4-even subgraph double cover.

Can we extend it to 5-even subgraph cover?

5-even subgraph double cover is NOT equivalent to 3-edge-coloring.

The Petersen graph is not 3-edge-colorable, but does have a 5-even subgraph double cover.

Conjecture (Jaeger)

Every bridgeless graph has a 5-even subgraph double cover.
How about 5-even subgraph double cover?

Equivalent statements:
(1) 3-edge-colorable;
(2) 3-even subgraph double cover;
(3) 4-even subgraph double cover.

Can we extend it to 5-even subgraph cover?

5-even subgraph double cover is NOT equivalent to 3-edge-coloring. The Petersen graph is not 3-edge-colorable, but does have a 5-even Subgraph double.

Conjecture (Jaeger)
Every bridgeless graph has a 5-even subgraph double cover.
How about 5-even subgraph double cover?

Equivalent statements:
(1) 3-edge-colorable;
(2) 3-even subgraph double cover;
(3) 4-even subgraph double cover.

Can we extend it to 5-even subgraph cover?
5-even subgraph double cover is NOT equivalent to 3-edge-coloring.
How about 5-even subgraph double cover?

Equivalent statements:
(1) 3-edge-colorable;
(2) 3-even subgraph double cover;
(3) 4-even subgraph double cover.

Can we extend it to 5-even subgraph cover?
5-even subgraph double cover is NOT equivalent to 3-edge-coloring.
The Petersen graph is not 3-edge-colorable, but does have a 5-even Subgraph double.

Conjecture (Jaeger) Every bridgeless graph has a 5-even subgraph double cover.
The Petersen graph
Circuit double cover conjecture:
Every bridgeless graph has a family of circuits that covers every edge precisely twice.
Circuit double cover conjecture:
Every bridgeless graph has a family of circuits that covers every edge precisely twice.

CDC conjecture is true for
- planar graphs
- graphs with strong 2-cell embedding on some surfaces
- 3-edge-colorable cubic graphs
Circuit double cover conjecture:
Every bridgeless graph has a family of circuits that covers every edge precisely twice.

CDC conjecture is true for
- planar graphs
- graphs with strong 2-cell embedding on some surfaces
- 3-edge-colorable cubic graphs
- graphs with nowhere-zero 4-flow (Jaeger)
Circuit double cover conjecture: 
Every bridgeless graph has a family of circuits that covers every edge precisely twice.

CDC conjecture is true for 
- planar graphs 
- graphs with strong 2-cell embedding on some surfaces 
- 3-edge-colorable cubic graphs 
- graphs with nowhere-zero 4-flow (Jaeger) 
- Cayley graphs (Hoffman, Locke and Meyerowitz)
Circuit double cover conjecture:
Every bridgeless graph has a family of circuits that covers every edge precisely twice.

CDC conjecture is true for
- planar graphs
- graphs with strong 2-cell embedding on some surfaces
- 3-edge-colorable cubic graphs
- graphs with nowhere-zero 4-flow (Jaeger)
- Cayley graphs (Hoffman, Locke and Meyerowitz)
- graphs with Hamilton path (Tarsi)
Circuit double cover conjecture:
Every bridgeless graph has a family of circuits that covers every edge precisely twice.

CDC conjecture is true for
- planar graphs
- graphs with strong 2-cell embedding on some surfaces
- 3-edge-colorable cubic graphs
- graphs with nowhere-zero 4-flow (Jaeger)
- Cayley graphs (Hoffman, Locke and Meyerowitz)
- graphs with Hamilton path (Tarsi)
- graphs without Petersen minor (Alspach, Goddyn, Z)