# So many conjectures, so little time: a warming-up survey

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# Contents

# Why we are here – apart from having fun

- The overall motivation is to continue the workshops of 1996–2008 in Enschede, Nectiny (twice), Hannover, Hajek and Domažlice in order to make **progress** on several intriguing conjectures.
- These highly related conjectures involve line graphs, claw-free graphs, cubic graphs, snarks, and concepts like Hamilton cycles, Hamilton-connectedness, dominating closed trails (circuits), and dominating cycles.
- Perhaps there is a link to double cycle covers and nowhere zero flows. We have specialists in these areas here as well.
- In order to introduce the workshop topics, I was asked to repeat some background, and present a **survey** on some of the conjectures and their relationships.

# The first two conjectures

The following two conjectures were tossed in the eighties.

Matthews & Sumner, 1984:

**Conjecture (MS-Conjecture)**

*Every 4-connected* **claw-free graph** *is hamiltonian.*

Thomassen, 1986:

**Conjecture (T-Conjecture)**

*Every 4-connected* **line graph** *is hamiltonian.*

Let me start by explaining the **terminology** to understand the above statements and their relationship.

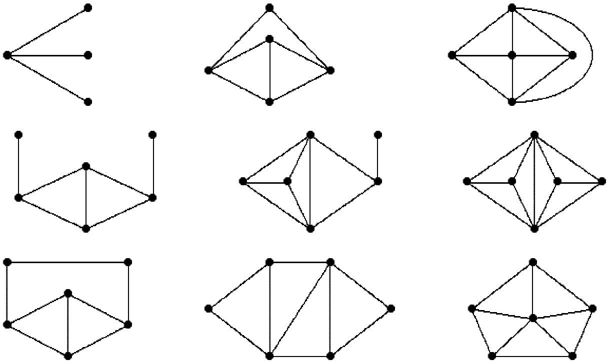
# The basics

- All graphs in this talk are finite, undirected, loopless and the majority is simple (in some results we allow multiple edges) .
- We denote a graph $G$ as $G = (V, E)$, where $V = V(G)$ is the vertex set and $E = E(G)$ is the edge set.
- A graph is *hamiltonian* if it contains a cycle through all its vertices, i.e., a connected spanning 2-regular subgraph.
- If $H$ is a graph, then the *line graph of* $H$, denoted by $L(H)$, is the graph on vertex set $E(H)$ in which two distinct vertices in $L(H)$ are adjacent if and only if their corresponding edges in $H$ share an end vertex (with a straightforward extension in case of multiple edges).
- A graph is a *line graph* if it is isomorphic to $L(H)$ for some graph $H$.
- Which graphs are line graphs and which are not?

# A characterization of line graphs

## Theorem (Beineke, 1969)

*A graph is a line graph if and only if it does not contain a copy of any of the following nine graphs as an* **induced** *subgraph.*



# Forbidden induced subgraphs

- Let $G$ be a graph and let $S$ be a nonempty subset of $V(G)$. Then the *subgraph of $G$ induced by $S$*, denoted by $G[S]$, is the graph with vertex set $S$, and all edges of $G$ with both end vertices in $S$.
- $H$ is an *induced subgraph* of $G$ if it is induced in $G$ by some subset of $V(G)$.
- $G$ is *$H$-free* if $H$ is not an induced subgraph of $G$.
- In particular, a graph $G$ is *claw-free* if $G$ does not contain a copy of the *claw* $K_{1,3}$ as an induced subgraph.
- Direct inspection of Beineke's result shows that every line graph is claw-free.

# The two conjectures revisited

**Conjecture (MS)**

*Every 4-connected claw-free graph is hamiltonian.*

**Conjecture (T)**

*Every 4-connected line graph is hamiltonian.*

- Since line graphs are claw-free the first conjecture is **stronger** than the second one.
- Or are they **equivalent**? (A question Herbert Fleischner posed during the EIDMA workshop on Hamiltonicity of 2-tough graphs, Hotel Hölterhof, Enschede, November 19-24, 1996.)

# A useful tool: the closure

- To answer the question affirmatively, Zdeněk Ryjáček introduced a **closure concept** for claw-free graphs at the same workshop.
- It is based on adding edges to a graph $G$ without destroying the (non)hamiltonicity (similar to the Bondy-Chvátal closure).
- The edges are added by looking at a vertex $v$ and the subgraph of $G$ induced by $N(v)$: the *neighbors* of $v$.
- If $G[N(v)]$ is connected and not a complete graph, all edges are added to turn $G[N(v)]$ into a complete graph.
- This procedure is repeated in the new graph, etc., until it is impossible to add any more edges.

# The two conjectures are equivalent

**Theorem (Ryjáček, 1997)**

*Let $G$ be a claw-free graph. Then*

- *the closure* $\mathrm{cl}(G)$ *is uniquely determined,*
- $\mathrm{cl}(G)$ *is hamiltonian if and only if* $G$ *is hamiltonian,*
- $\mathrm{cl}(G)$ *is the line graph of a triangle-free graph.*

**Corollary (using a result of Zhan, 1991)**

*Every 7-connected claw-free graph is hamiltonian.*

The conjectures are false for 3-connected graphs. There are other partial results for 6-connected graphs, but the best positive result to date is the following (The proof approach will be explained by Tomáš Kaiser).

**Theorem (Kaiser & Vrána, submitted)**

*Every* **5-connected** *claw-free graph with* **minimum degree at least 6** *is hamiltonian.*

# From line graphs to their root graphs

- Whenever we consider a line graph $G$, we can identify a graph $H$ such that $G = L(H)$ (in polynomial time).
- If $G$ is connected this $H$ is unique, except for $G = K_3$: then $H$ can be $K_3$ or $K_{1,3}$ (this is different for multigraphs).
- If we take $K_{1,3}$ in this exceptional case, we can talk of a unique $H$ as the *root graph* of the connected line graph $G$ isomorphic to $L(H)$.
- What is the counterpart in $H$ of a Hamilton cycle in $G$?
- A *closed trail* (circuit) is a connected eulerian subgraph, i.e., a connected subgraph in which all degrees are even.
- A *dominating closed trail* (DCT or D-circuit) is a closed trail $T$ such that every edge has at least one end vertex on $T$.

# Hamilton cycles and dominating closed trails

There is an intimate relationship between DCTs in $H$ and Hamilton cycles in $L(H)$.

**Theorem (Harary and Nash-Williams, 1965)**

*Let $H$ be a graph with at least three edges. Then $L(H)$ is hamiltonian if and only if $H$ contains a DCT.*

- What is the counterpart in $H$ of 4-connectivity in $L(H)$? Note that 4-edge-connectivity is not the right answer!
- A graph $H$ is *essentially 4-edge-connected* if it contains no edge-cut $R$ such that $|R| < 4$ and at least two components of $H - R$ contain an edge.
- $L(H)$ is 4-connected if and only if $H$ is essentially 4-edge-connected.

# Another equivalent conjecture

The previous results and observations imply that the following conjecture is **equivalent** to the two we have seen before.

**Conjecture (DCT-conjecture)**

*Every* **essentially** *4-edge-connected graph has a DCT.*

- Note that 4-edge-connected graphs contain two edge-disjoint spanning trees.
- Hence 4-edge-connected graphs contain a **spanning** closed trail, in particular a DCT.
- So line graphs of 4-edge-connected graphs are hamiltonian (and *Hamilton-connected*).

# From root graphs to cubic graphs

- If $H$ is *cubic*, i.e., 3-regular, then a DCT becomes a *dominating cycle* (abbreviated DC).
- A cubic graph is essentially 4-edge-connected if and only if it is cyclically 4-edge-connected.
- $H$ is *cyclically 4-edge-connected* if $H$ contains no edge-cut $R$ such that $|R| < 4$ and at least two components of $H - R$ contain a cycle.

**Conjecture (Ash & Jackson, 1984)**

*Every cyclically 4-edge-connected* **cubic** *graph has a DC.*

Fleischner and Jackson (1989) proved that this conjecture is **equivalent** to the others.

Main ingredient: Let $H$ be an essentially 4-edge-connected graph of minimum degree $\delta(G) \geq 3$ and let $v \in V(H)$ be of degree $d(v) \geq 4$. Then some *cubic inflation* of $H$ at $v$ is essentially 4-edge-connected.

# From cubic graphs to non-3-edge colorable cubic graphs

For *non-3-edge colorable* cubic graphs we have the following conjecture of Herbert Fleischner.

**Conjecture (F-Conjecture – not F-word)**

*Every cyclically 4-edge-connected cubic graph that is* **not 3-edge-colorable** *has a DC.*

Kochol (2000) proved that it is **equivalent** to the others, by assuming a counterexample to the previous one and constructing one to the F-Conjecture.

In 2002 he proved equivalence with seemingly weaker versions, with *sublinear defect*, e.g., the T-Conjecture is equivalent to:

**Conjecture (K-Conjecture)**

*There are sublinear functions $f_1(n)$ and $f_2(n)$ such that every 4-connected line graph $G$ of order $n$ contains $\leq f_1(n)$ paths that cover $\geq n - f_2(n)$ vertices of $G$.*

# From non-3-edge colorable cubic graphs to snarks

A *snark* is a cyclically 4-edge-connected cubic graph of **girth at least** 5 that is not 3-edge-colorable.

**Conjecture (Snark-Conjecture)**

*Every snark has a DC.*

The above conjecture is also **equivalent** to the others, as shown by B., Fijavž, Kaiser, Kužel, Ryjáček & Vrána (2008), using the constructive approach together with the concept of *contractible subgraphs*.

To date this is the **seemingly weakest** conjecture equivalent to the others.

Is there a link to the **Double Cycle Conjecture**?
Is there a link to **Nowhere Zero Flows**?

Let me turn to some **seemingly stronger** conjectures in the remainder of the talk.

# Seemingly stronger versions for cubic graphs

Fouquet & Thuillier (1990) established a seemingly stronger version than the Ash-Jackson-Conjecture.

It is stronger in the sense that they require a DC that contains two given disjoint edges, as follows.

**Conjecture**

*In a cyclically 4-edge-connected cubic graph* **any two disjoint edges** *are on a DC.*

The equivalence was extended by Fleischner & Kochol (2002) by requiring a DC through any two given edges.

**Conjecture**

*In a cyclically 4-edge-connected cubic graph* **any two edges** *are on a DC.*

# Seemingly stronger versions for cubic graphs

There are several further equivalent versions involving **subgraphs** of cubic graphs. Let me present them without going into technical details.

**Conjecture (Kužel, 2008)**

*Any subgraph $H$ of an essentially 4-edge-connected cubic graph with $\delta(H) = 2$ and $|V_2(H)| = 4$ is $V_2(H)$-dominated.*

**Conjecture (Kužel, Ryjáček & Vrána, to appear)**

*Any subgraph $H$ of an essentially 4-edge-connected cubic graph with $\delta(H) = 2$ and $|V_2(H)| = 4$ is strongly $V_2(H)$-dominated.*

**Conjecture (B., Fijavž, Kaiser, Kužel, Ryjáček & Vrána, 2008)**

*Every cyclically 4-edge-connected cubic graph contains a weakly A-contractible subgraph $F$ with $\delta(F) = 2$.*

# Back to line graphs – seemingly stronger versions

A graph is *Hamilton-connected* if there is a Hamilton path between any two vertices.

Kužel & Xiong (2004) established equivalence with the following conjecture.

**Conjecture**

*Every 4-connected line graph of a multigraph is* **Hamilton-connected**.

Ryjáček & Vrána (to appear) extended the equivalence to claw-free graphs.

**Conjecture**

*Every 4-connected claw-free graph is Hamilton-connected.*

# A link to the P versus NP problem

At present the seemingly strongest version of the conjectures is by Kužel, Ryjáček & Vrána (to appear).

A graph $G$ is *1-Hamilton-connected* if for any vertex $x$ of $G$ there is a Hamilton path in $G - x$ between any two vertices. They also define a slightly stronger property called *2-edge-Hamilton-connectivity* (more details in a later talk).

**Conjecture**

*Every 4-connected line graph of a multigraph is* **1-Hamilton-connected** *(2-edge-Hamilton-connected).*

This version strongly suggests that Thomassen's Conjecture (and all equivalent versions) might fail.

# How close are we to refuting the conjectures?

If the above conjecture is true, it implies that a line graph is 1-Hamilton-connected (2-edge-Hamilton-connected) **if and only if** it is 4-connected.

The connectivity of a (line) graph can be determined in **polynomial time**.

It is an **NP-complete** problem to decide whether a **line graph** is hamiltonian (Bertossi, 1981).

It is not difficult to show that deciding whether a given **graph** is 1-Hamilton-connected is also NP-complete.

It seems very **likely** that deciding whether a given graph is 1-Hamilton-connected remains NP-complete when restricted to line graphs. If we can show this, it implies that

*Thomassen's Conjecture cannot be true, unless P=NP.*

# How close are we to proving the conjectures?

The gap between the conjecture(s) and the positive results is narrowing.

If we drop the connectivity condition of the 2-regular spanning subgraph, we move from a Hamilton cycle to a *2-factor*. Enomoto, Jackson, Katerinis & Saito (1985) proved that every *2-tough* graph contains a 2-factor. This implies:

**Theorem**

*Every 4-connected claw-free graph has a* **2-factor***.*

It does not seem easy to use this as a starting point to show that there is a 2-factor with only one component, although there are some results that give upper bounds on the number of components. These results are beyond the scope of this talk.

# Relaxing the 4-connectedness and adding something else

I do not want to discuss degree conditions in this talk, but here is a connectivity-only result.

If we add an 'essentially connectivity' condition there is this result due to Lai, Shao, Wu & Zhou (2006).

**Theorem**

*Every 3-connected,* **essentially 11-connected** *claw-free (line) graph is hamiltonian.*

Perhaps 11 can be replaced by 5, which would be best possible (by the line graph of the Petersen graph in which the edges of a perfect matching are subdivided exactly once).

Question: how far can we decrease the 11 by raising the 3 to 4 in the theorem?

# Restrictions on the root graph

Lai (1994) proved the following partial affirmative answer to Thomassen's Conjecture.

**Theorem**

*Every 4-connected line graph of a planar graph is hamiltonian.*

Kriesell (2001) proved a similar result on line graphs of claw-free (multi)graphs with the stronger conclusion of Hamilton-connectedness. In fact, he proved the following more general result.

**Theorem**

*Let $G$ be a graph such that $L(G)$ is 4-connected and every vertex of degree 3 in $G$ is on an edge of multiplicity at least 2 or on a triangle of $G$. Then $L(G)$ is Hamilton-connected.*

# Restrictions on the root graph

Lai, Shao & Zhan (2004) did something similar for *quasi claw-free* graphs, i.e., in which every pair of vertices $u$ and $v$ at distance 2 has a common neighbor $w$ the neighbors of which are in $N[u] \cup N[v]$.

**Theorem**

*Every 4-connected line graph of a* **quasi claw-free** *graph is Hamilton-connected.*

There are many results along these lines.

In most proofs the root graphs are considered and the aim is to find a (closed or open) trail (internally) dominating all edges. A common approach is the following.

First the degree 1 vertices are deleted, then the degree 2 vertices are suppressed, and now one tries to show that the reduced graph has a suitable spanning (closed) trail. This is very similar to using two edge-disjoint spanning trees, or collapsibility, or advanced closure concepts.

# By the end of this week

We will have the following results:

- we will have improved on the currently best result that 5-connected claw-free graphs with $\delta \geq 6$ are hamiltonian, or
- we will have proved that 1-Hamilton-connectedness is NP-complete for line graphs, or
- we will have shown that all conjectures are equivalent to the famous Double Cycle Cover Conjecture, or
- ... this one is up to you!

I wish us all good luck.

And then there is coffee, the **brown** motor for all mathematicians!!